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Nonlinear r-modes in a spherical shell: issues of principle

Yuri Levin[(1),(2)] and Greg Ushomirsky[(2)]
[(1)] *Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125*
[(2)] *Department of Astronomy, 601 Campbell Hall, University of California, Berkeley, California 94720*

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ABSTRACT
We use a simple physical model to study the nonlinear behaviour of the r-mode instability. We assume that r-modes (Rossby waves) are excited in a thin spherical shell of rotating incompressible fluid. For this case, exact Rossby wave solutions of arbitrary amplitude are known. We find that:
(a) These nonlinear Rossby waves carry ZERO physical angular momentum and positive physical energy, which is contrary to the folklore belief that the r-mode angular momentum and energy are negative. We think that the origin of the confusion lies in the difference between physical and canonical quantities.
(b) Within our model, we confirm the differential drift reported by Rezzolla, Lamb and Shapiro (1999).
Radiation reaction is introduced into the model by assuming that the fluid is electrically charged; r-modes are coupled to electromagnetic radiation through current (magnetic) multipole moments. We study the coupled equations of charged fluid and Maxwell field dynamics and find that:
(c) To linear order in the mode amplitude, r-modes are subject to the CFS instability, as expected.
(d) Radiation reaction decreases the angular velocity of the shell and causes differential rotation (which is distinct from but similar in magnitude to the differential drift reported by Rezzolla et al.) prior to saturation of the r-mode growth. This is contrary to the phenomenological treatments to date, which assumed that, prior to the saturation of the r-mode amplitude, the loss of stellar angular momentum is accounted for by the r-mode growth. This establishes, for the first time, that radiation reaction leads not only to overall loss of angular momentum, but also to differential rotation.
(e) We show that for $l = 2$ r-mode electromagnetic radiation reaction is equivalent to gravitational radiation reaction in the lowest post-Newtonian order. Based on our electromagnetic calculations, we conclude that inertial frame dragging, both from the background rotation and from the r-mode itself, will modify the r-mode frequency by a factor $\sim R_{\rm Schwarzschild}/R_{\rm star}$, in qualitative agreement with Kojima (1998).

1 INTRODUCTION

Andersson (1998) has shown, and Friedman and Morsink (1998) have confirmed analytically, that r-modes of rotating stars can grow because of gravitational radiation reaction. Lindblom et al. (1998) have shown that this instability can be important in rapidly rotating hot neutron stars, where the r-mode amplitude might become large enough to affect the spin frequency [see also Andersson et al. (1998)]. The details of nonlinear evolution, which allows for large r-mode amplitudes, are essential for astrophysical applications [Owen et al. (1998), Spruit(1999), Levin (1999)]; yet only phenomenological treatments of such nonlinear evolution exist so far. This paper is an attempt to learn about the nonlinear behaviour of rotating fluid in which r-modes are driven by radiation reaction.

To address issues of principle, we choose to study a very simple system. Our "star" is a thin rotating shell of incompressible inviscid fluid which is sandwiched between two hard spheres. These spheres exert no friction on the fluid; their role is to make sure that the fluid motion is restricted to a two-dimensional spherical surface.

For such thin rotating shell, exact Rossby wave solutions[⋆] of the fully nonlinear fluid equations are known from the geophysical literature [see, e.g., Silberman (1954)]. In Section II we review these solutions and study their properties. We find that in our model Rossby waves carry zero physical angular momentum and positive energy. This is somewhat surprising, since Friedman and Morsink (1998) have shown that CANONICAL angular momentum and energy of the r-modes are negative. It seems that one cannot equate canonical and physical quantities for r-modes; one can only do it if the Lagrangian perturbation of vorticity is zero [Friedman and Schutz, 1978(a)]. This is not the case for

[⋆] From here onward "Rossby waves" and "r-modes" will be used interchangeably.

r-modes; this fact is intricately connected to the differential drift found by Rezzolla et al. (1999) and confirmed within our model. Our results do not argue against the presence of the CFS instability, since its derivation as given in Friedman and Schutz [1978(b)] relies only on canonical quantities.

In Section III we switch on radiation reaction by assuming that the fluid is electrically charged. We derive coupled dynamical fluid — Maxwell field equations [see Eqs (19)—(25)]. By studying the linear part of these equations, we then explicitly show the presence of the CFS instability for r-modes. We also show that the r-mode frequency is modified by radiation reaction; the frequency shift is given by $\Delta\omega \sim \gamma_{\rm CFS}(\lambda/R_{\rm star})^{2l+1}$, where l is the r-mode multipole order, $\gamma_{\rm CFS}$ is the r-mode growth rate and λ is the wavelength of the emitted gravitational wave [cf Eq. (33)].

We study the evolution of the star once the nonlinear coupling terms are included in the dynamical equations. We show that, to second order in the r-mode amplitude, radiation reaction slows down the star and causes it to rotate differentially, and that this is not related to the saturation of the r-mode. This is contrary to the phenomenological model introduced by Owen et al. (1998), and used extensively by Levin (1999). In that model it was assumed that, prior to saturation and in absence of viscosity, the loss of angular momentum to gravitational radiation is entirely accounted for by the r-mode growth. However, since (in our model) r-modes do not carry physical angular momentum, our finding only seems logical. The differential rotation is similar in magnitude, but different in origin from the differential drift found by Rezzolla et al. (1999). Both the slow-down and the differential rotation are described by Eq. (38) of the text.

Section IV provides a quick and somewhat superficial excursion into gravitational radiation reaction potential for mass current quadrupole moment, in the 3.5 post-Newtonian order. We use the formalism of Blanchet (1997) and others to show that in this order there is a full equivalence between the gravitational and electromagnetic radiation reaction. All of our results are thus expected to apply to the case of Rossby waves coupled to gravitational radiation, at least for $l = 2$. In particular, gravitational radiation reaction slows down the star and causes it to rotate differentially, prior to saturation of the r-mode growth. By carrying through results of Section III, we find that inertial frame dragging both from the background rotation and from the r-mode itself modifies the r-mode frequency by a factor of $\sim R_{\rm Schwarzschild}/R_{\rm star}$, in qualitative agreement with results of Kojima (1998).

2 ROSSBY WAVES IN A ROTATING SHELL: EXACT SOLUTIONS

Thin rotating shells, and Rossby waves in them, have been studied extensively by geophysicists and meteorologists since the end of last century. In this section we follow closely the work by Silberman (1954); more original references can be found in that work.

Consider a thin spherical sheet of incompressible fluid of radius a, sandwiched between two spherical hard covers, so that motion of the fluid is restricted to a two-dimensional spherical surface. The fluid is rotating with angular frequency Ω around the z-axis relative to an inertial observer. Since the fluid motion is restricted to two dimensions, and the fluid is assumed to be incompressible, the fluid velocity field in the co-rotating frame is completely determined by a stream function $\psi(\theta, \phi)$ defined on the fluid sphere:

$$v_\phi = \frac{1}{a}\frac{\partial\psi}{\partial\theta} \text{ and } v_\theta = -\frac{1}{a\sin\theta}\frac{\partial\psi}{\partial\phi}, \tag{1}$$

where v_ϕ and v_θ are the components of the fluid velocity along parallels and meridians, respectively. The vorticity of the fluid in the rotating frame of reference (called *relative* vorticity) is given by

$$\eta = (1/a)^2\nabla_a^2\psi, \tag{2}$$

where ∇_a^2 is the Laplacian operator on a unit sphere:

$$\nabla_a^2 = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3}$$

For inviscid, barotropic† flows, Euler's fluid equations imply conservation of circulation:

$$\int_{C(t)} \vec{v}^{\rm in}\cdot d\vec{r} = \text{const}, \tag{4}$$

where $C(t)$ is a contour moving with the fluid and $\vec{v}^{\rm in}$ is the velocity relative to an inertial observer. We derive the dynamical equations for an in incompressible spherical fluid shell from this equation. In particular, Eq. (4) implies that the radial component of the *absolute* vorticity is conserved:

$$\frac{d}{dt}(\eta + 2\Omega\cos\theta) = 0 \tag{5}$$

[see, e.g., Longuet-Higgins (1964)]. Here $d/dt = \partial/\partial t + \vec{v}\cdot\nabla$ is the Lagrangian time derivative. We would like to stress that Eq. (5) is fully nonlinear, and the only assumptions made so far are those of incompressibility and zero viscosity.

By expressing all velocity components and η in terms of the stream function [cf Eqs. (1) and (2)], one can write Eq. (5) as a dynamical nonlinear equation for the stream function evolution:

$$\frac{\partial}{\partial t}\nabla_a^2\tilde{\psi} + 2\Omega\frac{\partial}{\partial\phi}\tilde{\psi} = \frac{1}{\sin\theta}\{\nabla_a^2\tilde{\psi}, \tilde{\psi}\}, \tag{6}$$

where $\tilde{\psi} = \psi/a^2$ is the reduced stream function, and the brackets are defined by $\{A, B\} = \partial_\theta A\partial_\phi B - \partial_\theta B\partial_\phi A$. The left-hand side of Eq. (6) contains only linear terms, while the right-hand side is the nonlinear advection term.

Let us neglect this nonlinear part for a moment and look for a solution to the linearized equation in the form

$$\tilde{\psi} = \alpha\Omega Y_{lm}e^{i\omega t} + cc., \tag{7}$$

where α is a constant. It is then easy to work out the dispersion relation for the linear part of Eq. (6):

$$\omega = \frac{2m\Omega}{l(l+1)}. \tag{8}$$

This is the well-known dispersion relation for Rossby waves. Moreover, since Y_{lm} is an eigenfunction of the Laplacian, the nonlinear term in Eq. (6) equals zero when $\psi \propto Y_{lm}$.

† The exact condition for circulation conservation is $\nabla\rho\times\nabla p = 0$, where ρ and p are the fluid density and pressure respectively. A fluid with a one-parameter equation of state, $p = p(\rho)$, satisfies this condition automatically.

Therefore Eq. (7) is an *exact* solution of the fluid equations of motion, for arbitrarily large α.

Consider now the angular momentum and energy of the solution (7). The total angular momentum of the fluid shell is given by

$$L_{\rm star} = L_{\rm background} + L_{\rm r-mode}, \tag{9}$$

where $L_{\rm background}$ is the angular momentum of the shell when the r-mode amplitude α is zero, and

$$L_{\rm r-mode} = \rho a^3 \int v_\phi \sin^2\theta d\theta d\phi \tag{10}$$

is the physical change in angular momentum of the shell due to the r-mode. Here ρ is the surface density of the fluid. It is then clear from Eqs (1) and (7) that

$$L_{\rm r-mode} = 0 \tag{11}$$

for all $m \neq 0$. Modes with $m = 0$ have zero frequency and are called *zonal currents*; we will come back to them later when we discuss nonlinear evolution of the r-modes. On the other hand, the canonical angular momentum of a Rossby wave with $l = m$, computed from Eq. (3.4) of Owen et al. (1998), is

$$L_{\rm canonical} = -\frac{[l(l+1)]^2}{2}\rho a^4\alpha^2\Omega, \tag{12}$$

clearly different from the physical angular momentum.

The total kinetic energy of the shell ‡ is given by

$$E = \rho a^2 \int \left[(a\Omega\sin\theta + v_\phi)^2 + v_\theta^2\right]\sin\theta d\theta d\phi. \tag{13}$$

For r-modes with m not equal to zero, the terms linear in v on the right-hand side of Eq. (13) vanish after integration over ϕ. Therefore, the total energy of the star with the r-mode is

$$\begin{aligned} E &= E_{\rm background} + E_{\rm r-mode} \qquad (14)\\ &= \rho a^2 \int (a\Omega\sin\theta)^2 \sin\theta d\theta d\phi \\ &+ \rho a^2 \int \left(v_\theta^2 + v_\phi^2\right)\sin\theta d\theta d\phi. \end{aligned}$$

The physical r-mode energy $E_{\rm r-mode}$ is therefore greater than zero. Both this and $L_{\rm r-mode} = 0$ [see Eq. (11)] is contrary to the common belief, which holds that $E_{\rm r-mode}$ and $L_{\rm r-mode}$ are both negative.

The source of misunderstanding is the confusion between canonical and physical quantities, which are not equal to each other for r-modes. In their seminal work, Friedman and Schutz (1978a) have shown that canonical quantities are equal to the physical ones, so long as the Lagrangian change in vorticity is zero to second order in the perturbation amplitude. The last condition cannot be true for r-modes; this fact is intricately connected to the work by Rezzolla, Lamb and Shapiro (1999). These authors track the motion of a fluid particle for the case when the fluid stream function is given by $\psi \propto Y_{22}$. They find that the particle experiences a drift along stellar latitude; the speed of the drift depends on the latitude. Therefore, the authors argue, the presence of an r-mode implies the presence of a differential drift in the star§. This is clearly incompatible with the zero Lagrangian vorticity perturbations; therefore, the Friedman-Schutz condition is not satisfied and one cannot equate canonical and physical energy and angular momentum¶.

3 ELECTROMAGNETIC RADIATION REACTION FOR R-MODES

Friedman and Schutz (1978b) had shown that the CFS instability occurs whenever there is a mode with negative canonical angular momentum, which is dragged forward by the stellar rotation relative to an inertial observer, and which is coupled to some radiation field. This radiation field can be scalar, electromagnetic or gravitational—no matter what its nature, the CFS instability will be present. Papaloizou and Pringle (1978) investigated the CFS instability for f-modes coupled to a scalar field, and derived the growth rate of an unstable mode by introducing explicitly scalar-field radiation reaction.

In this section we introduce electromagnetic radiation reaction (ERR) into our model. In the next section we will show that, for r-modes on a spherical shell, gravitational radiation reaction (in the lowest appropriate post-Newtonian order) and ERR are equivalent. Thus, all results derived in this section for the case of the ERR are applicable to the case of gravitational radiation reaction.

We assume that the fluid is homogeneously electrically charged, with the charge q per unit mass‖. R-modes couple to electromagnetic radiation through time-varying current multipole moments (since the fluid is assumed to be incompressible, the charge multipole moments are zero). We now

‡ For an incompressible spherical fluid shell, both internal and potential energies are constant, and hence do not play any dynamical role.

§ In private communications, Rezzolla, Lamb, and Shapiro had found some of relativity community to be sceptical about the reality of their claimed differential drift. This scepticism was due to the fact that Rezzolla, Lamb, and Shapiro derived the differential drift using the fluid velocities which were first-order quantities in the r-mode amplitude; however, the drift that they found was second-order in the r-mode amplitude. The sceptics thought that such procedure was inconsistent. However, for the case of a spherical shell, the fluid velocities of the exact Rossby wave solution are STRICTLY linear with respect to the r-mode amplitude, and are given by exactly the same expressions as used by Rezzolla, Lamb, and Shapiro. Therefore, at least for the spherical shell, the differential drift due to the r-mode is real.

¶ One other obvious example where there is no equivalence between physical and canonical quantities is the shear sound wave in a solid elastic body. The physical motion of all the particles of the body is transverse to the direction of the wave propagation; therefore such wave has zero linear momentum along the direction of propagation. Yet, the canonical linear momentum is not zero (in fact, it is $p = N\hbar k$, where N is the number of phonons in the wave, and k is the wave vector).

‖ Note that this situation is different from the usual MHD, where the current is due to relative motion of oppositely charged species, e.g., electrons and protons. Here, the charge current density is proportional to the mass current density.

derive dynamical equations of motion of the fluid shell coupled to an electromagnetic field.

The circulation around a closed contour moving with the fluid is no longer conserved:

$$\frac{d}{dt}\int_{C(t)} \vec{v}^{\rm in}\cdot d\vec{r} = \int_{C(t)} \frac{d\vec{v}^{\rm in}}{dt}\cdot d\vec{r} + \int_{C(t)} \vec{v}^{\rm in}\cdot d\left(\frac{d\vec{r}}{dt}\right). \quad (15)$$

Here, as in Eq. (4), $\vec{v}^{\rm in}$ is the fluid velocity in the inertial frame of reference. The second term on the RHS of Eq. (15) is identically zero; the first one is in general not zero because of forces exerted on the fluid by the Maxwell field:

$$\begin{aligned}\frac{d}{dt}\int_{C(t)} \vec{v}^{\rm in}\cdot d\vec{r} &= q\int_{C(t)} \vec{E}\cdot d\vec{r} + \frac{q}{c}\int_{C(t)} \vec{v}^{\rm in}\times\vec{B}\cdot d\vec{r} \\ &= -\frac{q}{c}\frac{d}{dt}\int_{C(t)} \vec{B}\cdot d\vec{A}. \quad (16)\end{aligned}$$

Here $\int_{C(t)} \vec{B}\cdot d\vec{A}$ is the magnetic flux through the surface with boundary C, with $d\vec{A}$ being the area increment vector. Equation (16) is the well-known Faraday's law, which states that the electromotive force (EMF) around a closed contour equals to the negative of the rate of change of the magnetic flux through the contour. We see that the presence of a Maxwell field modifies Eq. (4) so that

$$\text{circulation} + (q/c)\cdot\text{flux} = \text{const}. \quad (17)$$

The analogue of Eq. (5) is then

$$\frac{d}{dt}(\eta + 2\Omega\cos\theta + \frac{q}{c}B_r) = 0, \quad (18)$$

where B_r is the radial component of the magnetic field in the inertial frame. Expressing velocity components in terms of the stream function derivatives [cf. Eq. (1)], we get

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_a^2\tilde{\psi} + 2\Omega\frac{\partial}{\partial\phi}\tilde{\psi} + \frac{q}{c}\frac{\partial B_r}{\partial t} &= \frac{1}{\sin\theta}\left\{\nabla_a^2\tilde{\psi},\tilde{\psi}\right\} \\ &+ \frac{q}{c\sin\theta}\left\{B_r,\tilde{\psi}\right\}. \quad (19)\end{aligned}$$

The angular operator ∇_a^2 is given by Eq. (3). Equation (19) governs the fluid motion under the action of an electromagnetic field, which, in turn, is generated by the fluid currents.

Let us consider a Rossby wave with a reduced stream function given by $\tilde{\psi}_{lm} = \tilde{\psi}_{lm}^{+} + \tilde{\psi}_{lm}^{-}$, where

$$\tilde{\psi}_{lm}^{+} = \left(\tilde{\psi}_{lm}^{-}\right)^{*} = \alpha\Omega Y_{lm}e^{i\omega_{lm}t}. \quad (20)$$

Then, as shown in Appendix A [cf. Eqs (A2), (A7), (A8), (A9), and (A10)], the radial component B_r of the magnetic field at the fluid shell is

$$B_r = B_{\rm rot} + B_{lm}, \quad (21)$$

where

$$B_{\rm rot} = \frac{8\pi q}{3c}\rho\Omega a\cos\theta \quad (22)$$

is the radial component of a dipole magnetic field due to the uniform rotation of the charged fluid shell, and

$$B_{lm} = \chi\tilde{\psi}_{lm}^{+} + \chi^{*}\tilde{\psi}_{lm}^{-} \quad (23)$$

is the radial component of the magnetic field produced by the Rossby wave itself. Here $\chi = \chi_1 + i\chi_2$, where, to lowest order in ka, we have

$$\chi_1 = -\frac{4\pi l(l+1)}{2l+1}\frac{q\rho a}{c}, \quad (24)$$

$$\chi_2 = \frac{4\pi l(l+1)}{[(2l+1)!!]^2}\frac{q\rho a}{c}(ka)^{2l+1}. \quad (25)$$

The wavevector k of the emitted electromagnetic waves is given by

$$k = \frac{1}{c}(\omega_{lm} - m\Omega) = -\frac{m}{l}\frac{(l-1)(l+2)}{l+1}\frac{\Omega}{c}. \quad (26)$$

We now obtain the new dispersion relation for Rossby waves by substituting Eq. (21) into Eq. (19) and keeping terms which are linear in the mode amplitude α. The resulting linear equation is

$$\frac{\partial}{\partial t}\nabla_a^2\tilde{\psi}_{lm} + 2\Omega\frac{\partial}{\partial\phi}\tilde{\psi}_{lm} + \frac{q}{c\sin\theta}\left\{\tilde{\psi}_{lm}, B_{\rm rot}\right\} + \frac{q}{c}\frac{\partial B_{lm}}{\partial t} = 0. \quad (27)$$

Using Eqs (22), (23), (24) and (25), we derive the Rossby-wave dispersion relation [cf. Eq. (8)]:

$$\omega_{lm} = \frac{2\Omega m}{l(l+1)}\frac{1+(1/3)\epsilon}{1+\epsilon/(2l+1) - i\epsilon(ka)^{2l+1}/[(2l+1)!!]^2}, \quad (28)$$

where

$$\epsilon = \frac{4\pi q^2\rho a}{c^2} = \frac{Q^2/a}{Mc^2}. \quad (29)$$

Here Q and M are the charge and the mass of the fluid shell respectively; ϵ is thus the ratio of the energy of electrostatic self-interaction and the rest-mass energy of the shell. For the case when a Rossby wave is coupled to gravitational radiation, we will see that the analogue of ϵ is $\sim R_{\rm Schwarzschild}/R_{\rm star}$ in the weak gravity regime. For a neutron star, $\epsilon \sim 0.4$. Then, to first order in ϵ, the angular frequency of the r-mode is

$$\omega_{lm} = \frac{2\Omega m}{l(l+1)}\left[1 + \frac{\Delta\omega_{lm}}{\omega_{lm}} - i\frac{\gamma_{\rm CFS}}{\omega_{lm}}\right], \quad (30)$$

where

$$\Delta\omega_{lm} = \frac{2(l-1)}{3(2l+1)}\epsilon\omega_{lm} \quad (31)$$

is the frequency shift of the r-mode due to the electromagnetic interaction, and

$$\gamma_{\rm CFS} = -\frac{\epsilon(ka)^{2l+1}}{[(2l+1)!!]^2}\omega_{lm} \quad (32)$$

is the growth rate of the r-mode due to the CFS instability. The angular frequency shift and the CFS growth rate are related by

$$\frac{\gamma_{\rm CFS}}{\Delta\omega_{lm}} \sim \left(\frac{\Omega a}{c}\right)^{2l+1} \sim \left(\frac{a}{\lambda}\right)^{2l+1}, \quad (33)$$

where λ is the wavelength of the emitted radiation.

Now we consider the terms which are of second order with respect to the r-mode amplitude α in the dynamical equation (19). The advection term (first term on RHS of Eq. [19]) does not contribute to this order, but the second term on the RHS of the Eq. (19) does have a component which is proportional to α^2:

$$\begin{aligned}\text{nonlinear term} &= \frac{q}{c\sin\theta}\left\{B_{lm},\tilde{\psi}_{lm}\right\} \quad (34)\\ &= \frac{q}{c\sin\theta}\left\{\chi\tilde{\psi}_{lm}^{+} + \chi^{*}\tilde{\psi}_{lm}^{-}, \tilde{\psi}_{lm}^{+} + \tilde{\psi}_{lm}^{-}\right\}\end{aligned}$$

$$= \quad 2i\alpha^2\Omega^2 Im(\chi)\frac{q}{c\sin\theta}\left\{Y_{lm}, Y_{lm}^*\right\}.$$

Let us focus, for concreteness, on the $l = m = 2$ mode, which has the largest growth rate (and hence is perhaps the most important astrophysically). Then the nonlinear term in Eq. (35) becomes

$$\text{nonlinear term} = 18\alpha^2\Omega\gamma_{\rm CFS}\left(\sqrt{\frac{9}{7\pi}}Y_{30} - \sqrt{\frac{3}{\pi}}Y_{10}\right). \quad (35)$$

This term on the right-hand side of Eq. (19) will act as a source for the left-hand side of this equation, thus creating a contribution to the stream function which is second order in α:

$$\tilde{\psi}_{(2)} = \kappa_1(t)Y_{10} + \kappa_2(t)Y_{30}, \quad (36)$$

where, to leading order in ϵ, κ_1 and κ_2 satisfy the following evolution equations:

$$\begin{aligned}\frac{d\kappa_1}{dt} &= 9\sqrt{\frac{3}{\pi}}\alpha^2\Omega\gamma_{\rm CFS},\\ \frac{d\kappa_2}{dt} &= -\frac{9}{2}\sqrt{\frac{1}{7\pi}}\alpha^2\Omega\gamma_{\rm CFS}.\end{aligned} \quad (37)$$

If the mode amplitude grows exponentially starting from a small value α_0, i.e., $\alpha = \alpha_0 e^{\gamma_{\rm CFS}t}$, then by integrating Eq. (37) we get the following expression for the reduced stream function of the radiation reaction induced flow:

$$\tilde{\psi}_{(2)} \simeq \frac{9}{2}\sqrt{\frac{3}{\pi}}\alpha^2\Omega Y_{10} - \frac{9}{4}\sqrt{\frac{1}{7\pi}}\alpha^2\Omega Y_{30}. \quad (38)$$

Let us discuss this equation. The first term on the right-hand side represents a uniform flow in the direction opposite to stellar rotation. *It thus represents the spindown of the star.* The angular momentum associated with this flow is

$$L_{\rm spindown} = -18\rho a^4\alpha^2\Omega, \quad (39)$$

in agreement with the canonical angular momentum of the r-mode itself [cf. Eq. (12)]. This is to be expected, as both $L_{\rm canonical}$ and $L_{\rm spindown}$ must be equal to the angular momentum lost to radiation.

The second term in Eq. (38) is a zonal current or, in another words, differential rotation. This term does not contribute to the angular momentum of the star. This differential rotation is similar in magnitude to the differential drift reported by Rezzolla et al. (1999). However, its origin is completely different. In our case, the differential rotation is driven by the radiation reaction, whereas the differential drift of Rezzolla et al. is a kinematic property of the r-mode fluid motion, and is not at all related to radiation reaction.

Note that both the uniform slow-down and the induced differential rotation are not related directly to the r-mode saturation, and are present well before saturation takes place. This is somewhat contrary to the phenomenological model of nonlinear behaviour of the r-mode instability by Owen et al. (1998), which was used extensively by Levin (1999). This model assumed that prior to r-mode saturation, the loss of angular momentum and energy carried off by gravitational waves was manifested by the r-mode growth, and that the background motion of the star was unchanged. The intuition for this model was based heavily on the belief that r-modes carry negative *physical* energy and angular momentum. We now know that the latter is in general not true, so it should not be surprising that radiation reaction induces second-order changes in the background motion of the star, *as well as* drives the r-mode instability. Astrophysical implications of this point are currently under investigation, and will be the subject of our next publication.

4 GRAVITATIONAL RADIATION REACTION FOR $L = 2$ R-MODES

An analogy between weak gravity and electromagnetism has been studied by many researchers [e.g., Braginsky, Caves and Thorne (1977), Thorne, Price and Macdonald (1986)]. Shapiro (1996) has shown that the Newtonian circulation around a closed contour comoving with a perfect fluid is not conserved in presence of a gravitomagnetic field; the conserved quantity (termed "relativistic circulation" by Shapiro) is a linear combination of the Newtonian circulation around the contour and a gravitomagnetic flux through the contour [cf. Eq. (4) of Shapiro (1996)]. This is very reminiscent of our conclusions for the circulation of a charged fluid in the presence of magnetic field; in fact, the two derivations are almost identical. Since it is the circulation equation that determines the dynamics of our rotating shell, Shapiro's result is of great relevance to understanding this dynamics.

In this section we use results of Blanchet (1997), Shapiro (1996), Asada et al. (1997), and Rezzolla et al. (1998) to investigate the effect of gravitational radiation reaction on r-modes in a spherical fluid shell. We find that, for slow-motion systems, there is a close analogy with electromagnetic radiation reaction considered in the previous section.

Following Asada et al. (1997) and Rezzolla et al. (1998), we consider a $3+1$ splitting of spacetime. We write the square of the line element as

$$\begin{aligned}ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \quad (40)\\ &= (\alpha^2 - \beta_i\beta^i)c^2dt^2 + 2\beta_i c dt dx^i + \gamma_{ij}dx^i dx^j,\end{aligned}$$

where α and β^i are the lapse function and the shift vector respectively, and γ_{ij} are the spatial metric coefficients. For weakly gravitating ($R \gg R_{\rm Schwarzschild}$), slow-motion ($v \ll c$) sources in which mass currents produce gravitational radiation, one can choose a gauge such that it is the time-varying shift vector $\vec{\beta}$ that plays a dynamically important role, relative to all other perturbations of the metric. For a periodic mass-current quadrupole moment, the shift vector consists of two parts, $\vec{\beta} = \vec{\beta^{\rm gm}} + \vec{\beta^{\rm rr}}$, where

1. The first part is the usual gravitomagnetic vector; its leading term in v/c is given by

$$\vec{\beta^{\rm gm}}(\vec{r}) = -\frac{4G}{c^3}\int \sigma\frac{\vec{v}^{\rm in}(\vec{r'})}{|\vec{r}-\vec{r'}|}d^3r', \quad (41)$$

where $\vec{v}^{\rm in}$ is the fluid speed relative to an inertial observer, and σ is the mass volume density of the source [cf. Eq. (3.4) of Blanchet (1997) when $c = \infty$; Blanchet uses a vector potential $\vec{V} = -\vec{\beta}/4$].

2. The second part is responsible for the radiation reaction; it changes sign under time reversal. Its leading term in v/c is given by [cf. Eq. (3.66) of Blanchet (1997) and Eq. (17.6) of Rezzolla et al. (1998)]:

$$\beta_i = \frac{16G}{45c^8}\epsilon_{ijk}x_i x_j S_{kl}^{(5)}, \quad (42)$$

where

$$S_{ij} = \int d^3x \epsilon_{kl(i}x_{j)}x_k \sigma v_l^{\rm in} \tag{43}$$

is the mass-current quadrupole moment. Here the superscript (n) stands for d^n/dt^n, and the brackets around tensorial indices, (ij), indicate symmetrization over these indices (i.e. $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ for a tensor a_{ij}).

The effective force $\vec{F}$ per unit mass that this shift metric perturbation exerts on the fluid is given by

$$\sigma^{-1}\vec{F} = -c\frac{\partial\vec{\beta}}{\partial t} + c\vec{v^{\rm in}} \times \nabla \times \vec{\beta}, \tag{44}$$

cf. Eq. (12) of Rezzolla et al. (1998) and Eq. (1) of Shapiro (1996). Note that this expression for the gravitational force is equivalent the Lorentz force exerted by an electromagnetic field on moving charged fluid:

$$\sigma^{-1}\vec{F}_{\rm em} = -\frac{q}{c}\frac{\partial\vec{A}}{\partial t} + \frac{q}{c}\vec{v^{\rm in}} \times \nabla \times \vec{A}. \tag{45}$$

In Equations (44) and (45), the shift vector β is dynamically equivalent to $c^2 q\vec{A}$, where $\vec{A}$ is the electrodynamical vector potential.

In our discussion of the motion of charged fluid on a spherical shell, we have shown that it was the radial component of the magnetic field that entered the dynamical equations of the fluid motion. Likewise, an identical argument will work for the gravitational force given by Eq. (44). Therefore, the radial component of $\nabla \times \vec{\beta}$ enters the equations of motion of a gravitating fluid:

$$\partial_t \nabla_a \tilde{\psi} + 2\Omega\frac{\partial\tilde{\psi}}{\partial\phi} + c\frac{\partial b}{\partial t} = \frac{1}{\sin\theta}\left\{\nabla_a\tilde{\psi}, \tilde{\psi}\right\} + \frac{c}{\sin\theta}\left\{b, \tilde{\psi}\right\}, \tag{46}$$

where $b = (\nabla \times \vec{\beta})_r$.

Suppose that a single $l = 2$ Rossby wave is excited in a spherical shell with the surface mass density ρ, and that the wave's reduced stream function is given by $\tilde{\psi}_{2m} = \alpha\Omega Y_{2m} e^{i\omega_{2m}t}$. Then, as is shown in Appendix B, in the slow-motion approximation the radial component of the gravitomagnetic field generated by the fluid motion is given by

$$b = b_{\rm rot} + b_{2m}; \tag{47}$$

cf. Eq. (21). Here

$$b_{\rm rot} = -\frac{4}{3}\epsilon_{\rm grav}\frac{\Omega}{c}\cos\theta, \tag{48}$$

and

$$b_{2m} = \chi_{\rm grav}\tilde{\psi}_{2m}, \tag{49}$$

where, to lowest order in ka,

$$Re\,(\chi_{\rm grav}) = \frac{12\epsilon_{\rm grav}}{3c}, \tag{50}$$

$$Im\,(\chi_{\rm grav}) = \frac{48\epsilon_{\rm grav}}{225c}(ka)^5, \tag{51}$$

and

$$\epsilon_{\rm grav} = \frac{2GM}{c^2 a} = \frac{R_{\rm Schwarzschild}}{a}. \tag{52}$$

Equations (47), (48), and (49) have the same structure as the analogous equations for the electromagnetic case, cf. Eqs (21), (22), and (23). By following the same steps as in the electromagnetic case, we work out the dispersion relation for $l = 2$ Rossby waves interacting with gravity:

$$\omega_{2m} = \frac{2\Omega}{3}\left(1 + \frac{\Delta\omega_{2m}}{\omega_{2m}} - i\frac{\gamma_{\rm CFS}}{\omega_{2m}}\right), \tag{53}$$

where

$$\Delta\omega_{2m} \simeq -\frac{4}{15}\epsilon_{\rm grav}\omega_{2m}, \tag{54}$$

is the shift of the r-mode frequency due to inertial frame dragging, which originates both from stellar rotation and from the mode itself; and

$$\gamma_{\rm CFS} \simeq -\omega_{2m}\frac{cIm\,(\chi_{\rm grav})}{6} = \omega_{2m}\frac{8\epsilon_{\rm grav}|ka|^5}{225} \tag{55}$$

is the growth rate of the r-mode due to the CFS instability driven by the gravitational radiation reaction. This growth rate agrees with the calculations of Lindblom et al. (1998) when one applies their Eq. (17) to the case of a massive spherical shell.

The r-mode frequency shift due to inertial frame dragging was discovered by Kojima (1998). For the case of a real three-dimensional star, Kojima (1998) claims, and Beyer and Kokkotas (1999) confirm, that such shift causes the r-mode spectrum to be continuous. This claim, however, is not supported by calculations of Lockitch, Andersson and Friedman (1999). The issue of a continuous spectrum is not relevant for the spherical shell.

Since the formalisms for a slow-motion gravitational radiation reaction from a mass-current quadrupole and for an electromagnetic radiation reaction from a charge-current quadrupole are identical in the structure of the dynamical equations, all of the conclusions from the previous section about the nonlinear electromagnetically-driven evolution of the $l = 2$ r-mode are also valid for a gravitationally driven $l = 2$ r-mode, at least for the case when the r-modes are excited in a spherical shell. More specifically, Eq. (38) for the secondary reaction-induced flow is still valid. Therefore, gravitational radiation reaction will slow down the star and cause it to rotate differentially; the former will account for the loss of angular momentum to gravitational waves. Both the slow-down of the star and the reaction-induced differential rotation are not related to the nonlinear saturation of the r-mode growth.

5 CONCLUSIONS AND BETS

This paper has studied the issues of principle for the r-mode instability in the nonlinear regime. Although we considered only a special case of r-modes excited in a spherical rotating shell, we bet that most of the lessons learned from the simple model will apply to real stars. In particular, r-modes in general do not carry negative *physical* angular momentum and energy; the radiation reaction causes the star to slow down and rotate differentially prior to the r-mode saturation. Therefore, phenomenological nonlinear evolution Equations (3.14) — (3.17) of Owen et al. (1998) need to be reconsidered.

We believe thus that our conclusions argue in favor of Spruit's conjecture (1999) that the r-mode instability causes differential rotation in the star. Spruit has modeled differential rotation as relative motion of two spherical shells, while we find differential rotation in the lateral direction. We believe that, in the three-dimensional case, both radial and lateral differential rotation will develop. A more detailed discussion of this and other astrophysical consequences of our formalism is a subject of a future publication.

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APPENDIX A1: MAGNETIC FIELD GENERATED BY A ROSSBY WAVE IN A CHARGED FLUID.

In this Appendix we find the radial component of the magnetic field generated by a Rossby wave in a charged fluid. We thus derive Eqs (21)—(25) of the text.

Suppose that a single Rossby wave is excited in a rotating shell, and that its reduced stream function is given by

$$\tilde{\psi}_{lm} = \tilde{\psi}^+_{lm} + \tilde{\psi}^-_{lm} = \alpha\Omega Y_{lm} e^{i\omega_{lm}t} + \alpha^*\Omega Y^*_{lm} e^{-i\omega_{lm}t}. \quad (A1)$$

The radial component of the magnetic field produced by such fluid motion can be found by using the multipole formalism discussed in Sec. 16.5 of Jackson (1975). In particular, using Eq. (16.87) of this reference, we find:

$$B_r = B_{\rm rot} + B_{lm}, \quad (A2)$$

where

$$B_{\rm rot} = \frac{q\rho}{c}\int \frac{2\Omega\cos\theta}{|\vec{r}-\vec{r'}|} a^2 \sin\theta d\theta d\phi, \quad (A3)$$

and

$$B_{lm} = \frac{q\rho}{ac}\int \frac{e^{-ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}\vec{r'}\cdot\nabla\times\vec{v} a^2 \sin\theta' d\theta' d\phi'. \quad (A4)$$

Here $B_{\rm rot}$ is the dipole magnetic field due to the uniform rotation of the charged shell, while B_{lm} is the field due to the Rossby wave. The wavenumber k is that of the emitted electromagnetic radiation:

$$k = \frac{1}{c}(\omega_{lm} - m\Omega). \quad (A5)$$

Equations (A3) and (A4) are evaluated by noting that $\vec{r}\cdot\nabla\times\vec{v} = a\nabla^2_a\tilde{\psi}_{lm} = -al(l+1)\tilde{\psi}^+_{lm} + c.c.$, and that

$$\int \frac{e^{-ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|} Y_{lm}(\theta',\phi')\sin\theta' d\theta' d\phi' \quad (A6)$$
$$= 4\pi ik h^{(1)}_l(ka) j_l(ka) Y_{lm}(\theta,\phi),$$

where j_l and $h^{(1)}_l$ are spherical Bessel and Hankel functions respectively (see e.g. Eqs (16.9) and (16.10) of Jackson (1975)). After some algebra, we get

$$B_{\rm rot} = \frac{8\pi q}{3c}\rho\Omega a\cos\theta, \quad (A7)$$

and

$$B_{lm} = \chi\tilde{\psi}^+_{lm} + \chi^*\tilde{\psi}^-_{lm} \quad (A8)$$

Here $\chi = \chi_1 + i\chi_2$, where, to lowest order in ka,

$$\chi_1 = -\frac{4\pi l(l+1)}{2l+1}\frac{q\rho a}{c}, \quad (A9)$$

$$\chi_2 = \frac{4\pi l(l+1)}{[(2l+1)!!]^2}\frac{q\rho a}{c}(ka)^{2l+1}. \quad (A10)$$

Thus, we have derived Equations (21), (22), (23), (24), and (25) of the text.

APPENDIX A2: GRAVITOMAGNETIC FIELD GENERATED BY AN $L = 2$ ROSSBY WAVE IN A THIN SHELL

In this Appendix we derive Eqs (47), (49), (50), and (51) for the radial component of the gravitomagnetic field, $b = (\nabla\times\vec{\beta})_r$, generated by a rotating massive shell in which an $l = 2$ Rossby wave is excited.

Evaluation of the part of b which is not responsible for radiation reaction, to leading order in v/c, is straightforward. The relevant part of the shift vector, $\vec{\beta}_{\rm nonradiative}$, is given by Eq. (41) of the text, which, up to a constant factor, is same as the nonradiative part of the electromagnetic vector potential:

$$\vec{A}_{\rm nonradiative} = \frac{1}{c}\int \frac{\sigma q\vec{v}^{\rm in}(r')}{|r-r'|} d^3r'. \quad (A11)$$

Therefore,

$$\vec{\beta}_{\rm nonradiative} = -\frac{4G}{c^2 q}\vec{A}_{\rm nonradiative},$$
$$b_{\rm nonradiative} = -\frac{4G}{c^2 q}B_{r\,\rm nonradiative}. \quad \text{(A12)}$$

Here, as in the text, σ is the mass *volume* density of the fluid (trivial changes must be made for the case of a two-dimensional sphere), and q is the charge per unit mass of the fluid. By using Eqs. (21), (22), (23), and (24) of the text, and substituting $l = 2$, we get

$$b_{\rm nonradiative} = -\frac{4}{3}\epsilon_{\rm grav}\frac{\Omega}{c}\cos\theta + \frac{12\epsilon_{\rm grav}}{3c}\tilde{\psi}_{2m}, \quad \text{(A13)}$$

where $\epsilon_{\rm grav} = 2GM/(c^2 a) = R_{\rm Schwarzschild}/a$.

Now we shall derive the expression for the part of the radial gravitomagnetic field which is responsible for radiation reaction; we shall denote it by $b^{\rm rr}$. The radiative part of the shift vector is given by Eq. (42) of the text, which can be rewritten as follows:

$$\beta_i^{\rm rr} = \left[\frac{8G}{45c^7}\epsilon_{ijk}x_j x_l \int \sigma\left(J'_k x'_l + J'_l x'_k\right) d^3x'\right]^{(5)}, \quad \text{(A14)}$$

or, in index-free form,

$$\vec{\beta}^{\rm rr} = \left\{\frac{8G}{45c^7}\int \sigma\left[\left(\vec{r}\cdot\vec{r}'\right)\left(\vec{r}\times\vec{J}'\right) + \left(\vec{r}\cdot\vec{J}'\right)\left(\vec{r}\times\vec{r}'\right)\right] d^3x'\right\}^{(5)}, \text{(A15)}$$

where

$$\vec{J}' = \vec{r}' \times \vec{v}(\vec{r}') = -\nabla\psi(\vec{r}'). \quad \text{(A16)}$$

In this expression, vectors $\vec{r}$, $\vec{r}'$, and the fifth time derivative are defined relative to an inertial observer at rest. The radial component of the radiation reaction gravitomagnetic field is given by

$$b^{\rm rr} = \frac{1}{a}(\vec{r}\cdot\nabla\times\vec{\beta}^{\rm rr}) = \left[\frac{48G\sigma}{45c^8 a}\int \sigma(\vec{r}\cdot\vec{r}')[\vec{r}\cdot\nabla\psi(\vec{r}')]d^3x'\right]^{(5)}, \text{(A17)}$$

or, for the case of a spherical shell,

$$b = \frac{48G\rho a}{45c^8}\left[\int(\vec{r}\cdot\vec{r}')[\vec{r}\cdot\nabla\psi(\vec{r}')]\sin\theta' d\theta' d\phi'\right]^{(5)}. \quad \text{(A18)}$$

Let the stream function ψ be a linear combination of $l = 2$ spherical harmonics:

$$\psi(\theta,\phi) = \sum_{m=-2}^{2} a_{2m}Y_{2m}(\theta,\phi). \quad \text{(A19)}$$

Let us choose a coordinate system x_1, y_1, z_1 (with polar angles θ_1 and ϕ_1), such that the axis z_1 is along $\vec{r}$. In this system, ψ is also a linear of $l = 2$ spherical harmonics, but with different weight coefficients a^1_{2m}:

$$\psi(\vec{r}_1) = \sum_{m=-2}^{2} a^1_{2m}Y_{2m}(\theta_1,\phi_1). \quad \text{(A20)}$$

For $\vec{r}_1 = \vec{r}$ (i.e. for $\theta_1 = 0$),

$$\psi(\vec{r}) = a^1_{20}Y_{20}(0,\phi_1) = \sqrt{\frac{5}{4\pi}}a^1_{20}. \quad \text{(A21)}$$

In the new coordinate system we can now evaluate $b(\vec{r})$:

$$b(\vec{r}) = \frac{48\pi G\rho a^3}{45c^7} \quad \text{(A22)}$$
$$\times \left\{\int \cos\theta_1\left[\vec{r}\cdot\nabla\sum_{m=-2}^{2}a^1_{2m}Y_{2m}(\theta_1,\phi_1)\right]\sin\theta_1 d\theta_1 d\phi_1\right\}^{(5)}.$$

In Eq. (A23), all terms with non-zero m vanish after integration over ϕ; only the term with $m = 0$ contributes to the integral. Integrating the remaining term, we get

$$b(\vec{r}) = \left[\frac{48G\rho}{45c^8}\frac{8}{5}\pi a^4\sqrt{\frac{5}{4\pi}}a^1_{20}\right]^{(5)}. \quad \text{(A23)}$$

Substituting Eq. (A21) into Eq. (A23), and evaluating the fifth time derivative, we get

$$b(\vec{r}) = i\frac{48}{225}\epsilon_{\rm grav}(ka)^5\frac{\tilde{\psi}_{2m}}{c}. \quad \text{(A24)}$$

Here $\epsilon_{\rm grav} = 2GM/(c^2 a) = R_{\rm Schwarzschild}/a$. Equations (A13) and (A24) are equivalent to Equations (47), (48), (49), (50), and (51) of the text.